Three Results in Connection with Inverse Matrices

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ABSTRACT

First we show that the Moore-Penrose solution of an arbitrary system of linear equations is a convex combination of the solutions of all uniquely solvable partial systems. The other two results concern the elements of inverse Toeplitz band matrices, namely the asymptotic behavior of a determinant appearing in a formula of D. S. Meek and a modification of a formula of W. D. Hoskins and P. J. Ponzo for matrices with binomial coefficients in the limit case.

1. A DETERMINANT REPRESENTATION FOR THE MOORE-PENROSE SOLUTION

Let $Ax = z$ be a linear system of equations with $A \in \mathbf{C}^{n,m}$, and $x^+ = A^+z$ the corresponding Moore-Penrose solution. According to G. Zielke [23] there exist several generalizations of Cramer's rule to the components of x^+ or, what is equivalent with this, several representations of the elements of A^+ as rational functions of certain determinants with elements from A, from E. H. Moore [14], to R. Gabriel [10], to J. Springer [16]. Cf. also M. Stojaković [18] and the references cited there, as well as the generalizations by G. C. Verghese [20] and H. J. Werner [21] of a formula of Ben-Israel. Though such formulas have mainly theoretical interest, J. Springer [17] has used their denominator for the foundation of a residue arithmetic to compute the Moore-Penrose inverse exactly. Here we begin with a new proof of one variant of the generalized Cramer's rule.

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With this aim we denote the rank of A by r and introduce the multiindices

$$p=(p_1,\dots,p_r),\qquad q=(q_1,\dots,q_r)$$

with $1\leqslant p_1<p_2<\cdots<p_r\leqslant n$ and $1\leqslant q_1<q_2<\cdots<q_r\leqslant m$. The notation $i\in q$ shall mean that $q_k=i$ for a certain k. Further, for $A=(a_{ij})$ we denote

$$_pA=\left(a_{p_ij}\right)\in\mathbb{C}^{r,m},\qquad A_q=\left(a_{iq_j}\right)\in\mathbb{C}^{n,r},\qquad {}_pA_q=\left(a_{p_iq_j}\right)\in\mathbb{C}^{r,r},$$

and finally, for a square matrix we use the notation $|A|=\det A$ for the determinant, $\|A\|=|\det A|$ for the modulus of the determinant, $\bar{A}=(\bar{a}_{ij})$, and $A_i[z]$ for the matrix A with its ith column replaced by the vector z, the last notation making sense also for rectangular matrices.

THEOREM 1. *The ith component of the Moore-Penrose solution x of $Ax=z$ possesses the representation*

$$x_i^+=\frac{\sum'_{p,q}|{}_p\bar{A}_q||{}_pA_{iq}[{}_pz]|}{\sum_{p,q}\|{}_pA_q\|^2},\tag{1}$$

where the double sums run over all possible multiindices p,q, but with the prime indicating $i\in q$ in the numerator sum.

Proof. In the case $n=m=r$ there exists only the trivial possibility $p=q=(1,\dots,r)$, so that ${}_pA_q=A$, and in view of $\|A\|^2=|\bar{A}||A|$ the formula (1) is just Cramer's rule.

In the general case we can write (cf. G. Zielke [22])

$$A=BC\qquad\text{with}\quad B\in\mathbb{C}^{n,r},\quad C\in\mathbb{C}^{r,m},$$

so that $A^+=C^+B^+$ with $B^+=(B^*B)^{-1}B^*$, $C^+=C^*(CC^*)^{-1}$. Hence the system $Ax=z$ splits up into the systems $By=z$, $Cx=y$, and we can calculate the Moore-Penrose solutions $y^+=B^+z$, $x^+=C^+y^+$ separately. In the special case $n=m$ we make use of the multiplication theorem (cf. W. I. Smirnow [15])

$$|CB|=\sum_p|C_p||{}_pB|.\tag{2}$$

1. From the representation $y^+ = (B^*B)^{-1}B^*z$ we find, in view of $(B^*B)_i[B^*z] = B^*(B_i[z])$ and Cramer's rule,

$$y_i^+ = \frac{|B^*B_i(z)|}{|B^*B|},$$

and according to (2) we obtain

$$y_i^+ = \frac{\sum\limits_p |B_p^*||_pB_i[_pz]|}{|B^*B|}. \tag{3}$$

Using once more (2) with $C = B^*$ as well as $|B_p^*| = |_p\bar{B}|$, we have proved (1) in the special case $A = B$, because for q there exists only the trivial possibility.

2. From $x^+ = C^*(CC^*)^{-1}y$ we find for the ith component $x_i^+ = \bar{c}_{\cdot i}(CC^*)^{-1}y$, denoting by $\bar{c}_{\cdot i}$ the vector of the ith row of C^*. Applying the identity of Magnus (cf. C. Brezinski [9]), we obtain

$$x_i^+ = -\frac{1}{|CC^*|}\left|\begin{pmatrix} 0 & \bar{c}_{\cdot i} \\ y & CC^* \end{pmatrix}\right|.$$

Now (2) yields

$$\left|\begin{pmatrix} 0 & \bar{c}_{\cdot i} \\ y & CC^* \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & \delta_{i\cdot} \\ y & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & C^* \end{pmatrix}\right| = \sum_{q^0}\left|\begin{pmatrix} 0 & \delta_{i\cdot} \\ y & C \end{pmatrix}_{q^0}\right|\left|_{q^0}\begin{pmatrix} 1 & 0 \\ 0 & C^* \end{pmatrix}\right|,$$

where $q^0 = (q_0, q_1, \ldots, q_r)$, $0 \leqslant q_0 < q_1 < \cdots < q_r \leqslant m$, and $\delta_{i\cdot}$ denotes the ith row of the r-dimensional unit matrix. In the case $q_0 = 0$ we have

$$\left|_{q^0}\begin{pmatrix} 1 & 0 \\ 0 & C^* \end{pmatrix}\right| = |_qC^*|;$$

otherwise the determinant on the left-hand side vanishes. In the case $q_0 = 0$ and $i \in q^0$ we have

$$\left|\begin{pmatrix} 0 & \delta_{i\cdot} \\ y & C \end{pmatrix}_{q^0}\right| = -|C_{iq}[y]|,$$

whereas in the case $i \notin q^0$ the matrix on the left-hand side vanishes. Hence we obtain the result

$$x_i^+ = \frac{\sum_q' |C_{iq}[y]|\,|_qC^*|}{|CC^*|}, \tag{4}$$

where the prime on the sum indicates that there only q with $i \in q$ are allowed. In the special case $A = C$ and $z = y$ this is, according to (2) with $B = C^*$, just (1), because for p there exists only the trivial possibility.

3. In the general case, since $x^+ = C^+ y^+$, we have to substitute (3) into (4) with $y = y^+$. Introducing the vector $z^{(p)}$ with the components $|_pB_i[_pz]|$ for $i = 1, \ldots, r$, we obtain

$$x_i^+ = \frac{\sum_{p,q}' |B_p^*|\,|C_{iq}[z^{(p)}]|\,|_qC^*|}{|B^*B|\,|CC^*|}. \tag{5}$$

Since $_pA_q = (_pB)(C_q)$, we have $|_pA_q| = |_pB|\,|C_q|$, so that in view of

$$\sum_{p,q} \| _pA_q \|^2 = \sum_p \| _pB \|^2 \sum_q \| C_q \|^2 \tag{6}$$

and (2), the denominator of (5) equals the denominator of (1). In the regular case the formula $(_pA_q)^{-1} = (C_q)^{-1}(_pB)^{-1}$ implies, by Cramer's rule, $_pA_{iq}[_pz] = C_{iq}[z^{(p)}]$, and by continuity arguments this result must also be valid in the singular case. Hence in view of $|_p\bar{A}_q| = |_qA_p^*| = |_qC^*|\,|B_p^*|$, Equation (5) goes over into (1), and the theorem is proved. ■

REMARKS. The double sum in the numerator of (1) contains $\binom{n}{r}$ terms in p and $\binom{m-1}{r-1}$ terms in q, i.e. altogether

$$\binom{n}{r}\binom{m-1}{r-1}$$

terms. The double sum in the denominator contains

$$\binom{n}{r}\binom{m}{r}$$

terms.

To *interpret* the result we introduce the notations

$$\beta=\sum_p \|{}_pB\|^2, \qquad \beta_p=\frac{1}{\beta}\|{}_pB\|^2, \qquad \gamma=\sum_q \|C_q\|^2, \qquad \gamma_q=\frac{1}{\gamma}\|C_q\|^2.$$

In the case $|{}_pA_q|\neq 0$ let $x^{(p,q)}$ be the canonical imbedding of the solution of ${}_pA_qx={}_pz$ into the m-dimensional space, this means that $x^{(p,q)}$ possesses according to Cramer's rule the components

$$x_i^{(p,q)}=\frac{|{}_pA_{iq}[{}_pz]|}{|{}_pA_q|}$$

for $i\in q$, and $x_i^{(p,q)}=0$ otherwise. In the singular case we define $x^{(p,q)}$ to be the zero vector. Now in view of (5) and (6) we can write (1) in the form

$$x_i^+=\sum_{p,q}{}' \beta_p\gamma_q x_i^{(p,q)}. \tag{7}$$

The definition of $x^{(p,q)}$ in the singular case $|{}_pA_q|=0$ does not influence the components (7) in view of $\beta_p\gamma_q=0$. Since

$$\sum_p \beta_p=1, \qquad \sum_q \gamma_q=1,$$

we obtain from (7) the

COROLLARY. *The Moore-Penrose solution x^+ of the linear system $Ax=z$ is the convex combination*

$$x^+=\sum_{p,q}\beta_p\gamma_q x^{(p,q)} \tag{8}$$

of the solutions of all uniquely solvable r-dimensional subsystems canonically imbedded into $\mathbb{C}^m$.

Note that the sum in (8) has, in contrast to (7), no prime. In the case $m=r$ with trivial q and $\gamma_q=1$, where we can put $C=I$, i.e. $A=B$, the components (7) can be interpreted as the *least-squares solutions of the linear systems*

$$|{}_pA|x_i=|{}_pA_i[{}_pz]|$$

for all p, but a fixed i.

2. TOEPLITZ MATRICES

In this paragraph we summarize some notation and results needed later on. A square Toeplitz matrix $A \in \mathbb{C}^{n,n}$ possesses the structure

$$A = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-p} & & \bigcirc \\ a_1 & a_0 & \cdot & & \ddots & \\ \vdots & \cdot & \cdot & \cdot & & a_{-p} \\ a_q & & \cdot & \cdot & \cdot & \vdots \\ & \ddots & & \cdot & a_0 & a_{-1} \\ \bigcirc & & a_q & \cdots & a_1 & a_0 \end{pmatrix}. \tag{9}$$

We assume $1 \leqslant p, q < n$ and $a_{-p}a_q \neq 0$. Generally speaking, A is a band matrix, though the case $p = q = n-1$ is allowed. Because of their simple shape and their importance for many applications, there exists an extensive literature on these matrices, from which we cite only A. Böttcher and B. Silbermann [8], G. Heinig and K. Rost [11], and L. Berg [5].

The linear system $Ax = z$ is equivalent to the difference equation

$$a_{-p}x_{i+p} + \cdots + a_0 x_i + \cdots + a_q x_{i-q} = z_i \tag{10}$$

for $i = 1, \dots, n$ under the boundary conditions

$$x_0 = x_{-1} = \cdots = x_{1-q} = 0, \qquad x_{n+1} = \cdots = x_{n+p} = 0. \tag{11}$$

Let h_i be the solution of the homogeneous equation belonging to (10) under the initial conditions

$$h_p = 1/a_{-p}, \qquad h_i = 0 \quad \text{for } i < p. \tag{12}$$

Then the *two-sided infinite Toeplitz matrix* $H = (h_{i-j})$, $-\infty < i, j < \infty$, is an inverse of the two-sided infinite matrix corresponding to (9). Let

$$a_{-p}\lambda^{p+q} + \cdots + a_0\lambda^q + \cdots + a_q \tag{13}$$

be the characteristic polynomial of (10), and $\lambda_{-r}, \dots, \lambda_s$ the pairwise different zeros of (13) with the multiplicities $k_{-r},\dots,k_s$ such that $k_{-r}+\cdots+k_s = p+q$. Then h_i possesses for $i>-q$ the representation

$$h_i = \sum_{\rho=-r}^{s} P_\rho(i)\lambda_\rho^i, \tag{14}$$

which can be constructed with the help of operational calculus, where the coefficients $P_\rho(i)$ are polynomials in i with the exact degree $k_\rho - 1$.

Now we assume that $A\in\mathbf{C}^{n,n}$ is regular, and we introduce the *inverse matrix* $A^{-1}=(d_{ij})$, $i,j=1,\dots,n$. Among several well-known explicit formulas for the elements d_{ij} we only mention the recent formula of D. S. Meek [13] (cf. also L. Berg [4])

$$d_{ij} = \frac{1}{\Delta_n}\begin{vmatrix} h_{i-j} & h_{n+1-j} & \cdots & h_{n+p-j} \\ h_i & h_{n+1} & \cdots & h_{n+p} \\ \vdots & \vdots & & \vdots \\ h_{i+p-1} & h_{n+p} & \cdots & h_{n+2p-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} h_{n+1} & h_{n+2} & \cdots & h_{n+p} \\ h_{n+2} & h_{n+3} & \cdots & h_{n+p+1} \\ \vdots & \vdots & & \vdots \\ h_{n+p} & h_{n+p+1} & \cdots & h_{n+2p-1} \end{vmatrix}. \tag{15}$$

For the determinant of A there is an explicit formula by W. F. Trench [19] (cf. also L. Berg [6]).

Finally, we consider the *one-sided infinite matrix* $A=(a_{i-j})$, $1\leqslant i,j<\infty$, with $a_i=0$ for $i>q$ and $i<-p$ corresponding to (9) under the condition

$$|\lambda_{-r}|\leqslant\cdots\leqslant|\lambda_0|<|\lambda_1|\leqslant\cdots\leqslant|\lambda_s| \tag{16}$$

with $k_{-r}+\cdots+k_0=q$, $k_1+\cdots+k_s=p$. Then operational calculus provides an inverse matrix $G=(g_{ij})$ of A with the elements

$$g_{ij} = \sum_{\rho=-r}^{0}\sum_{\mu=0}^{k_\rho-1} b_{\rho\mu}\sum_{\sigma=1}^{s}\sum_{\nu=0}^{k_\sigma-1} c_{\sigma\nu}G_{ij}^{(\mu,\nu)}(\lambda_\rho,\lambda_\sigma), \tag{17}$$

where $b_{\rho\mu}$, $c_{\sigma\nu}$ are the coefficients of the decompositions into partial fractions

$$\begin{aligned} \lambda^{q-1}\prod_{\rho=-r}^{0}(\lambda-\lambda_\rho)^{-k_\rho} &= \sum_{\rho=-r}^{0}\sum_{\mu=0}^{k_\rho-1}\frac{\lambda^\mu b_{\rho\mu}}{(\lambda-\lambda_\rho)^{\mu+1}},\\ a_{-p}\prod_{\sigma=1}^{s}(\lambda-\lambda_\sigma)^{-k_\sigma} &= \sum_{\sigma=1}^{s}\sum_{\nu=0}^{k_\sigma-1}\frac{c_{\sigma\nu}}{(\lambda-\lambda_\sigma)^{\nu+1}}, \end{aligned} \tag{18}$$

and where

$$\begin{aligned} G_{ij}^{(\mu,\nu)}(\alpha,\beta) &= \frac{1}{\mu!\nu!}\frac{\partial^{\mu+\nu}}{\partial\alpha^\mu\partial\beta^\nu}\frac{1}{\beta-\alpha}\\ &\quad\times\begin{cases}(\alpha^{i+\mu}\beta^{-j}-\alpha^\mu\beta^{i-j}) & \text{for} \quad i\leqslant j,\\ (\alpha^{i+\mu}\beta^{-j}-\alpha^{i+\mu-j}) & \text{for} \quad i\geqslant j.\end{cases} \end{aligned} \tag{19}$$

3. THE ASYMPTOTIC BEHAVIOR OF MEEK'S DETERMINANT

The knowledge of the asymptotic behavior of inverse Toeplitz matrices for $n\to\infty$ is useful for the estimation of the numerical stability of large linear systems with such matrices [5]. Another application of these asymptotics is the construction of holomorphic solutions of partial differential equations [1], [7]. Here we only deal with Meek's determinant Δ_n from (15). At the end of this section we make some remarks concerning the elements d_{ij}.

We introduce the generalized Vandermonde matrix

$$V(\lambda)=(\lambda^i i^{j-1})=\begin{pmatrix}\lambda & \lambda & \cdots & \lambda\\ \lambda^2 & 2\lambda^2 & \cdots & 2^{k-1}\lambda^2\\ \vdots & \vdots & & \vdots\\ \lambda^p & p\lambda^p & \cdots & p^{k-1}\lambda^p\end{pmatrix}$$

with $i=1,\dots,p$ and $j=1,\dots,k$, and the square matrix

$$Q(\lambda)=\left(\frac{1}{(i-1)!(j-1)!}P^{(i+j-2)}(n)\lambda^n\right) \tag{20}$$

with $i, j = 1,\ldots,k$. Then for a polynomial $P(n)$ of degree $k-1$, Taylor's expansion

$$P(n+i+j)\lambda^{n+i+j} = \sum_{\nu,\mu=1}^{k} \lambda^{i} i^{\nu-1} \frac{1}{(\nu-1)!(\mu-1)!} P^{(\nu+\mu-2)}(n)\lambda^{n+j} j^{\mu-1}$$

is equivalent to the matrix equation

$$\left(P(n+i+j)\lambda^{n+i+j}\right) = V(\lambda)Q(\lambda)V^{T}(\lambda)$$

with $i, j = 1,\ldots,p$. According to (14) we have

$$\left(h_{n+i+j}\right) = \left(\sum_{\rho=-r}^{s} P_{\rho}(n+i+j)\lambda_{\rho}^{n+i+j} \right).$$

With the notation $V_{\rho} = V(\lambda_{\rho})$ with $k = k_{\rho}$, $Q_{\rho} = Q(\lambda_{\rho})$ with $P(n) = P_{\rho}(n)$, and the block matrices $V = (V_{s},\ldots,V_{-r}) \in \mathbb{C}^{p,p+q}$, $Q = \operatorname{diag}(Q_{s},\ldots,Q_{-r}) \in \mathbb{C}^{p+q,p+q}$, this equation reads

$$\left(h_{n+i+j}\right) = VQV^{T}. \tag{21}$$

THEOREM 2. *Under the assumption* (16) *with* $k_{1} + \cdots + k_{s} = p$, *one has the asymptotic representation*

$$\Delta_{n} \sim \delta\left(\lambda_{s}^{k_{s}} \cdots \lambda_{1}^{k_{1}}\right)^{n} \tag{22}$$

for $n \to \infty$ *with a certain constant* $\delta \neq 0$.

Proof. The matrix $Q(\lambda)$ from (20) possesses on the secondary diagonal $i + j = k + 1$ the elements

$$\frac{1}{(i-1)!(j-1)!} P^{(k-1)}(n)\lambda^{n}$$

and below this diagonal only zero elements. Since $P^{(k-1)}(n)$ is a constant independent of n, we find for the determinant of $Q(\lambda)$ the form

$$|Q(\lambda)| = \gamma\lambda^{kn}$$

with $\gamma \neq 0$. From (15) and (21) we obtain

$$\Delta_{n+1} = |VQV^T|,$$

and this determinant does not change its value if we add to the rows of V linear combinations of the other rows of V. By such manipulations we can transform V into a matrix $\tilde{V}$ with zero elements below the main diagonal. Since $\tilde{V} \in \mathbb{C}^{p,p+q}$ and $k_1 + \cdots + k_s = p$, this means that in the corresponding block representation $\tilde{V} = (\tilde{V}_s, \ldots, \tilde{V}_1, \tilde{V}_0, \ldots, \tilde{V}_{-r})$ the partial matrix $(\tilde{V}_s, \ldots, \tilde{V}_1)$ is a square upper triangular matrix with a determinant different from zero, since $|V_s, \ldots, V_1|$ is a generalized Vandermondian. Now it is easily seen that for $n \to \infty$

$$\begin{aligned}\Delta_{n+1} = |\tilde{V}Q\tilde{V}^T| &\sim |V_s, \ldots, V_1|^2 |\operatorname{diag}(Q_s, \ldots, Q_1)| \\ &= \delta\left(\lambda_s^{k_s} \cdots \lambda_1^{k_1}\right)^{n+1},\end{aligned}$$

since the elements of $Q_0, \ldots, Q_{-r}$ possess, in view of (16) and (20), a lower order than the other elements. Hence the assertion (22) is proved. ■

Now we consider the case

$$|\lambda_{-r}| \leqslant \cdots \leqslant |\lambda_0| < |\lambda_1| = \cdots = |\lambda_t| < |\lambda_{t+1}| \leqslant \cdots \leqslant |\lambda_s|$$

with

$$p = l + k_{t+1} + \cdots + k_s, \qquad 1 \leqslant l < k_1 + \cdots + k_t$$

and $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_t$. The quadratic form

$$R = \sum_{\rho=1}^{t} (k_\rho - \beta_\rho)\beta_\rho \tag{23}$$

may have a unique maximum for integer variables β_ρ under the conditions $0 \leqslant \beta_\rho \leqslant k_\rho$ and

$$\beta_1 + \cdots + \beta_t = l.$$

The explicit solution of this *quadratic optimization problem* is given in [3].

THEOREM 3. *Under the assumptions just mentioned one has the asymptotic representation*

$$\Delta_n \sim \delta n^R \left(\lambda_1^{\beta_1} \cdots \lambda_t^{\beta_t} \lambda_{t+1}^{k_{t+1}} \cdots \lambda_s^{k_s}\right)^n \tag{24}$$

for $n \to \infty$ with a certain constant $\delta \neq 0$, where $\beta_1, \dots, \beta_t$ is the solution of the optimization problem and R the corresponding maximal value of (23).

Proof. We only sketch the changes to the foregoing proof. For $\rho = 1, \dots, t$ we consider the block decomposition

$$Q_\rho = \begin{pmatrix} X_\rho & W_\rho \\ U_\rho & Z_\rho \end{pmatrix}$$

with $X_\rho \in \mathbb{C}^{\beta_\rho, \beta_\rho}$, $Z_\rho \in \mathbb{C}^{k_\rho - \beta_\rho, k_\rho - \beta_\rho}$, and substitute in Q the partial block $\operatorname{diag}(Q_t, \dots, Q_1)$ by the rearrangement

$$\begin{pmatrix} X & W \\ U & Z \end{pmatrix}$$

with $X = \operatorname{diag}(X_t, \dots, X_1)$ and analogous definitions of U, W, Z. Analogously, we split up $V_i = (V_i^+, V_i^-)$, where V_i^+ consists of the β_i first and V_i^- of the $k_i - \beta_i$ last columns of V_i, and rearrange V as $(V_s, \dots, V_{t+1}, V_t^+, \dots, V_1^+, V_t^-, \dots, V_1^-, V_0, \dots, V_{-r})$. Then it follows as before that

$$\Delta_{n+1} \sim |V_s, \dots, V_{t+1}, V_t^+, \dots, V_1^+|^2 |\operatorname{diag}(Q_s, \dots, Q_{t+1}, X_t, \dots, X_1)|$$

and from this, since

$$|X_\rho| = \gamma_\rho n^{(k_\rho - \beta_\rho)\beta_\rho} \lambda_\rho^{\beta_\rho n}$$

with $\gamma_\rho \neq 0$, we have the assertion (24). ∎

REMARKS. Under the assumptions of Theorem 2 we expect

$$d_{ij} = \begin{cases} O\big((j - i + p)^{k_+ - 1} \lambda_1^{i-j}\big) & \text{for } \ i < j + p, \\ O\big((i - j + q)^{k_- - 1} \lambda_0^{i-j}\big) & \text{for } \ i > j - q \end{cases} \tag{25}$$

uniformly for $1 \leqslant i, j \leqslant n$ and $n \to \infty$, with $k_+ = \max k_\rho$ for $|\lambda_\rho| = |\lambda_1|$ and $k_- = \max k_\rho$ for $|\lambda_\rho| = |\lambda_o|$. Under the assumptions of Theorem 3 we have according to [3]

$$d_{ij} = O\big((i \wedge j) i^{k_t - \beta_t - 1} j^{\beta_t - 1} \lambda_1^{i-j}\big)$$

uniformly for $1 \leqslant i, j \leqslant n$ and $n \to \infty$ with $i \wedge j = \min(i, j)$. There exist examples in which the last estimation can be sharpened to

$$d_{ij}=O\left((i\wedge j)i^{k_t-\beta_t-1}j^{\beta_t-1}\left(1-\frac{i\wedge j}{n+1}\right)\right.$$
$$\left.\times\left(1-\frac{i}{n+1}\right)^{\beta_t-1}\left(1-\frac{j}{n+1}\right)^{k_t-\beta_t-1}\lambda_1^{i-j}\right), \tag{26}$$

but a general proof of (25) and (26) is still to be found.

4. THE INVERSE OF A TOEPLITZ MATRIX WITH BINOMIAL COEFFIC1ENTS

First we generalize (19) to the case $\alpha=\beta$. For this purpose we write (19) in the form

$$G_{ij}^{(\mu,\nu)}(\alpha,\beta)=\varepsilon\frac{1}{\mu!\nu!}\frac{\partial^{\mu+\nu}}{\partial\alpha^\mu\partial\beta^\nu}\alpha^k\beta^l\frac{\beta^m-\alpha^m}{\beta-\alpha}$$

with $k=\mu$, $l=-j$, $m=i$, $\varepsilon=-1$ for $i\leqslant j$, and $k=i+\mu$, $l=0$, $m=-j$, $\varepsilon=1$ for $i\geqslant j$ or, what is the same,

$$G_{ij}^{(\mu,\nu)}(\alpha,\beta)=\frac{\varepsilon m}{\mu!\nu!}\frac{\partial^{\mu+\nu}}{\partial\alpha^\mu\partial\beta^\nu}\alpha^k\beta^l\int_0^1[\alpha+(\beta-\alpha)u]^{m-1}du.$$

Applying the Leibniz formula, we obtain

$$G_{ij}^{(\mu,\nu)}(\alpha,\beta)=\frac{\varepsilon m}{\mu!\nu!}\sum_{\rho=0}^{\mu}\sum_{\sigma=0}^{\nu}\binom{\mu}{\rho}\binom{\nu}{\sigma}(k)_{\mu-\rho}\alpha^{k-\mu+\rho}(l)_{\nu-\sigma}\beta^{l-\nu+\sigma}$$
$$\times(m-1)_{\rho+\sigma}\int_0^1[\alpha+(\beta-\alpha)u]^{m-\rho-\sigma-1}(1-u)^\rho u^\sigma du$$

with $(k)_{\mu-\rho}=k(k-1)\cdots(k-\mu+\rho+1)$ and analogous expressions in the other cases. For $\alpha=\beta$ we find, since $k+l+m=\mu+i-j$,

$$G_{ij}^{(\mu,\nu)}(\beta,\beta)=\varepsilon\beta^{i-j-\nu-1}\sum_{\rho=0}^{\mu}\sum_{\sigma=0}^{\nu}\binom{k}{\mu-\rho}\binom{l}{\nu-\sigma}\binom{m}{\rho+\sigma+1}.$$

According to the elementary formula

$$\sum_{\rho=0}^{\mu}\binom{\mu}{\rho}\binom{i}{\rho+\sigma+1}=\binom{i+\mu}{\mu+\sigma+1}$$

we have therefore the result

$$G_{ij}^{(\mu,\nu)}(\beta,\beta)$$

$$=\begin{cases}\beta^{i-j-\nu-1}\sum_{\sigma=0}^{\nu}(-1)^{\nu+\sigma+1}\binom{j+\nu-\sigma-1}{\nu-\sigma}\binom{i+\mu}{\mu+\sigma+1} & \text{for}\quad i\leqslant j,\\ \beta^{i-j-\nu-1}\sum_{\rho=0}^{\mu}(-1)^{\nu+\rho+1}\binom{j+\nu+\rho}{\nu+\rho+1}\binom{i+\mu}{\mu-\rho} & \text{for}\quad i\geqslant j.\end{cases} \tag{27}$$

The formula (27) possesses a *symmetry property*. To prove this we denote the sums in (27) by $G_{ijk}^{(\mu,\nu)}$ with $k=1$ in the first and $k=2$ in the second case, and consider the generating function

$$\sum_{i=1}^{\infty}G_{ij1}^{(\mu,\nu)}u^{i-1}=\sum_{\sigma=0}^{\nu}(-1)^{\nu+\sigma+1}\binom{j+\nu-\sigma-1}{\nu-\sigma}\frac{u^{\sigma}}{(1-u)^{\mu+\sigma+2}}.$$

In view of the fact that

$$\left(\frac{u}{1-u}\right)^{\sigma}=\left(\frac{1}{1-u}-1\right)^{\sigma}=\sum_{\rho=0}^{\sigma}\binom{\sigma}{\rho}\frac{(-1)^{\sigma+\rho}}{(1-u)^{\rho}}$$

and the elementary formula

$$\sum_{\sigma=\rho}^{\nu}\binom{j+\nu-\sigma-1}{\nu-\sigma}\binom{\sigma}{\rho}=\binom{j+\nu}{\nu-\rho},$$

we find

$$\begin{aligned}\sum_{i=1}^{\infty}G_{ij1}^{(\mu,\nu)}u^{i-1}&=\sum_{\rho=0}^{\nu}(-1)^{\nu+\rho+1}\binom{j+\nu}{\nu-\rho}\frac{1}{(1-u)^{\mu+\rho+2}}\\&=\sum_{i=1}^{\infty}(-1)^{\mu+\nu}(-1)^{\mu+\nu}G_{ji2}^{(\nu,\mu)}u^{i-1}\end{aligned}$$

and therefore obtain the desired identity

$$(-1)^{\mu}G_{ij}^{(\mu,\nu)}(1,1)=(-1)^{\nu}G_{ji}^{(\nu,\mu)}(1,1). \tag{28}$$

A concrete example is

$$T=\left((-1)^{i+j+p}\binom{p+q}{i-j+p}\right) \tag{29}$$

with $i, j = 1,2,\ldots$. In this case we consider the polynomial $(\lambda-1)^{p+q}$ from (13) as the limit case of $(\lambda-\lambda_1)^p(\lambda-\lambda_0)^q$ for $\lambda_0 \to \lambda_1 = 1$. Hence in (18) we have $r=0$, $k_0=q$, $s=1$, $k_1=p$, and $b_{0,q-1}=c_{1,p-1}=1$ with $b_{0\mu}=b_{1\nu}=0$ otherwise, and the formula (17) for the elements of an inverse matrix of (29) implies

$$g_{ij}=G_{ij}^{(q-1,p-1)}(1,1). \tag{30}$$

Let us mention that W. D. Hoskins and P. J. Ponzo [12] have given a formula in the case $p=q$ for the elements of the inverse of T in the n-dimensional case, which for $n\to\infty$ converges to

$$g_{ij}=\binom{i}{p}\sum_{k=0}^{p-1}(-1)^{k+1}\binom{p}{k}\binom{j+k+p-1}{2p-1}\frac{p-k}{i+k}$$

for $i \geqslant j$ and $g_{ji}=g_{ij}$. Other representations may be found in [2]. But all these formulas are less convenient than the representation (30) with (27).

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